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Incompressible slip flow past a semi-infinite flat plate

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An asymptotic solution to the Navier–Stokes equation is obtained for the incompressible flow of a viscous fluid past a semi-infinite flat plate when a slip boundary condition is applied at the plate. The results for the shear stress (and hence the slip velocity) on the plate differ basically from those obtained by previous authors who considered the same problem using some form of the Oseen equations.

1. Introduction

Many theoretical investigations have been made of the effect of a slip velocity on the flow of a viscous fluid past bodies. The problems of practical interest are generally in the hypersonic flow range. However, it is felt that a solution for an incompressible case may be of use because it is in the low Mach number range that experimental verification may be obtained. It is also of some academic interest in view of the divergence of opinion among authors as to the form of the correction to the Blasius term for the skin friction due to slip.

Mirels (1952) and Bell (1955) consider the linearized slip flow of an incompressible viscous fluid past a semi-infinite flat plate using a boundary-layer form of the Oseen equations. Laurmann (1961) considers the same steady problem but uses the full linearized Oseen equations. Using the Wiener–Hopf technique he obtains an asymptotic solution which differs from that obtained by Mirels (1952) and Bell (1955); the last two give the same result. Laurmann (1961) thus concludes that boundary-layer theory is inadequate in that it cannot predict the solution at or near the plate. He suggests that the solution to the Oseen equations for this type of problem will contain most of the essential features of the full solution to the Navier–Stokes equations. Their results for the local shear stress on the plate are given in § 3 for reference and comparison.

In this paper the first five terms of an asymptotic solution to the full Navier–Stokes equations are obtained for the steady viscous incompressible slip flow past a semi-infinite flat plate by extending the method and solution obtained by Goldstein (1956, 1960) and Murray (1965) for the no-slip case. This is a boundary-layer (or rather a singular perturbation) approach to the problem in the strict sense. The extension for higher-order terms is suggested. The shear stress on the plate (and thus the slip velocity) is found and compared with that found by Mirels (1952) and Laurmann (1961) and it is shown that the Oseen equations possibly do not predict the same behaviour on the plate as do the Navier–Stokes equations. The exact form for the solution, and hence the skin friction coefficient cannot be found completely at this stage, since the asymptotic solution is dependent on an undetermined constant, which allows this solution to be joined onto that valid near the leading edge. The same situation obtains in the no-slip case.

The solution is obtained in parabolic co-ordinates, the optimal co-ordinates for the semi-infinite flat plate configuration (see Kaplun 1954). These co-ordinates also alleviate to some extent the difficulties encountered at the leading edge when Cartesian co-ordinates are used; Mirels (1952), Bell (1955) and Laurmann (1961) use the latter. The use of parabolic co-ordinates specifically in the problem of slip flow past a semi-infinite flat plate was suggested by Goldstein (1956).

2. Differential equations and asymptotic solution

The problem considered is that of the flow of an incompressible viscous fluid, of kinematic viscosity ν, past a semi-infinite flat plate $y_1 = 0$, $x_1 \geqslant 0$, where x_1, y_1 are Cartesian co-ordinates. An accepted slip condition (see references given by Laurmann 1961) is

$$u_1 = \lambda \frac{\partial u_1}{\partial y_1} \quad (y_1 = 0, x_1 \geqslant 0), \tag{1}$$

where u_1 is the velocity in the x_1-direction and

$$\lambda = CM\nu/U, \quad C = 1{\cdot}26\gamma^{\frac{1}{2}}(2-\sigma)/\sigma, \tag{2}$$

where M is the Mach number, U the free stream velocity, γ the ratio of specific heats of the fluid and σ the plate reflexion coefficient. The role of M is that of a parameter only and does not imply that compressibility is taken into account.

Introduce parabolic co-ordinates ξ_1, η_1 by

$$\zeta_1^2 = (\xi_1 + i\eta_1)^2 = x_1 + iy_1, \tag{3}$$

where $\arg z_1 = 0, 2\pi$, on the upper and lower side of the plate respectively. This maps the $z_1 = (x_1 + iy_1)$-plane, cut along the real axis, onto the upper half of the ζ_1-plane and so $0 \leqslant \arg \zeta_1 \leqslant \pi$.

The Navier–Stokes equation for the stream function ψ_1 in parabolic co-ordinates ξ_1, η_1 is

$$\nu\left\{(\xi_1^2+\eta_1^2)\left(\frac{\partial^2\Delta_1}{\partial\xi_1^2}+\frac{\partial^2\Delta_1}{\partial\eta_1^2}\right)+4\left(\Delta_1-\xi_1\frac{\partial\Delta_1}{\partial\xi_1}-\eta_1\frac{\partial\Delta_1}{\partial\eta_1}\right)\right\}$$
$$+(\xi_1^2+\eta_1^2)\frac{\partial(\psi_1,\Delta_1)}{\partial(\xi_1,\eta_1)}+2\left(\xi_1\frac{\partial\psi_1}{\partial\eta_1}-\eta_1\frac{\partial\psi_1}{\partial\xi_1}\right)\Delta_1 = 0, \tag{4}$$

where
$$\Delta_1 = \frac{\partial^2\psi_1}{\partial\xi_1^2}+\frac{\partial^2\psi_1}{\partial\eta_1^2}.$$

The slip parameter λ must appear in the co-ordinates to obtain the appropriate asymptotic solution, and in the usual singular-perturbation manner dimensionless stretched variables ξ, η are introduced by

$$\eta = \left(\frac{U}{\nu}\right)^{\frac{1}{2}}\eta_1, \quad \xi = \frac{1}{\lambda}\left(\frac{\nu}{U}\right)^{\frac{1}{2}}\xi_1 = \frac{1}{CM}\left(\frac{U}{\nu}\right)^{\frac{1}{2}}\xi_1. \tag{5}$$

An asymptotic solution for large ξ is sought. The solution near the plate must merge into a potential solution for η large and it must do so with exponentially small vorticity. This potential flow must merge into the free stream as $\eta_1 \to \infty$. As in the zero-slip case the least general potential solution into which the boundary layer merges is given by $\psi_1 = \operatorname{Im} w$, where, excluding a constant,

$$w = -\zeta'^2+\beta\zeta'+\sum_{m=1}\frac{1}{\zeta'^m}[b_{m,m}(\log\zeta')^m+b_{m,m-1}(\log\zeta')^{m-1}+\ldots+b_{m,1}\log\zeta'+b_{m,0}], \tag{6}$$

where the $b_{i,j}$ are real and

$$\beta = 1\cdot 7208, \quad \zeta' = \zeta e^{-\frac{1}{2}i\pi} = \eta - i\xi$$

(see Goldstein 1960 for a discussion of this form). The expansion of (6) for large $\xi(|\xi| > \eta)$ suggests that with

$$\psi_1 = (U\nu)^{\frac{1}{2}}\xi_1 f(\xi,\eta) = CM\nu\xi f(\xi,\eta), \tag{7}$$

an appropriate asymptotic form of $f(\xi,\eta)$ in (7) with the boundary condition (1) in mind is

$$f(\xi,\eta) = f_0(\eta)+\frac{1}{2\xi}f_0'(\eta)+\frac{1}{\xi^2}[g_2(\eta)\log\xi+f_2(\eta)]+\frac{g_2'(\eta)\log\xi}{2\xi^3}+O\left(\frac{1}{\xi^3}\right), \tag{8}$$

where the prime denotes differentiation with respect to η. The form of the first two terms was suggested by Goldstein (1956) but the subsequent terms there are different. The third and fourth terms are suggested by the zero-slip solution (see Goldstein 1960 and Murray 1965).

From (1) in the co-ordinates given by (5) and the form of ψ_1 from (7), the boundary conditions at $\eta = 0$ for $f(\xi,\eta)$ are

$$f(\xi,0) = 0, \quad f_\eta(\xi,0) = \frac{1}{2\xi}f_{\eta\eta}(\xi,0), \tag{9}$$

the first of these ensuring that there is no flow across the plate. For ξ large and $|\xi| > \eta$ comparison of (8) with (6) shows that

$$f_0(\eta) \sim 2\eta-\beta, \quad f_2(\eta) \sim b_{10}, \quad g_2(\eta) \sim b_{11}, \tag{10}$$

where the error terms must be exponentially small.

Equation (4) is now written in terms of ξ and η as given by (5). Substitution of (7) and (8) into this equation and equating powers of ξ gives the ordinary differential equations for the f's and g's. The form of higher-order terms than given in (8) is clear when comparison is made with this form and the zero-slip case. Terms of $O(\xi^3)$ and also $O(\xi^2)$ result in

$$f_0'''+f_0 f_0'' = 0. \tag{11}$$

Terms of $O(\xi\log\xi)$ and $O(\log\xi)$ both give

$$L_2(g_2) = 0 \tag{12}$$

and of $O(\xi)$ give

$$L_2(f_2) = \frac{1}{(CM)^2}\left[\frac{d}{d\eta}(\eta f_0'-f_0)^2\right]+[g_2''f_0'-g_2f_0''']-\tfrac{1}{2}f_0''f_0''', \tag{13}$$

where the operator

$$L_2 = \frac{d^4}{d\eta^4} + f_0\frac{d^3}{d\eta^3} + 3f_0'\frac{d^2}{d\eta^2} + f_0''\frac{d}{d\eta} - f_0'''.$$

Equations (11) and (12) are of the same form as those for the corresponding zero-slip case (see Goldstein 1960 and Murray 1965) while (13) is the same in form if $CM = 1$ and $-\frac{1}{2}f_0''f_0'''$ is omitted from the right side.

The form of (8) and the boundary conditions (9) imply that the f's and g's must have *double* zeros at the origin, $\eta = 0$. Thus with conditions (10) the solution of (11), $f_0(\eta)$, is the Blasius function where

$$\left.\begin{aligned} &f_0 \sim 2\eta - \beta, \quad \beta = 1\cdot 7208, \quad f_0''(0) = \alpha = 1\cdot 32824, \\ &f_0'' \sim O(\exp -(\eta - \tfrac{1}{2}\beta)^2). \end{aligned}\right\} \tag{14}$$

The form of the g_2 of (12) with a double zero at $\eta = 0$ and which asymptotes to a constant b_{11} is given by Goldstein (1956) and

$$g_2(\eta) = b_{11}(\eta f_0' - f_0)/\beta, \tag{15}$$

where b_{11} is as yet undetermined.

Complementary functions of (13) with double zeros at $\eta = 0$ are, for small η,

$$\left.\begin{aligned} y_2^{(2)} &= \eta f_0' - f_0, \\ y_2^{(3)} &= \alpha\eta^3/3! - 8\alpha^2\eta^6/6! + \ldots, \end{aligned}\right\} \tag{16}$$

and for large η,

$$y_2^{(2)} \sim \beta, \quad y_2^{(3)} \sim \tfrac{1}{4}\alpha\eta + a_2^{(3)} + b_2^{(3)}E_2 + c_2^{(3)}H_2, \tag{17}$$

where $a_2^{(3)}$, $b_2^{(3)}$, $c_2^{(3)}$ are constants, none of which is zero (see Murray 1965) and where E_2 decays algebraically with leading term $(\eta - \frac{1}{2}\beta)^{-1}$ whilst H_2 is exponentially small. Denote by $y_2^{(4)}$ the particular integral of (13) with the first square bracket only on the right and $CM = 1$, $y_2^{(5)}$ that with the second square bracket and $b_{11} = \beta$, and $y_2^{(6)}$ that with $-\frac{1}{2}f_0''f_0'''$ only. Then, from (13)

$$\left.\begin{aligned} y_2^{(4)} &\sim \tfrac{1}{4}\beta^2\eta + a_2^{(4)} + b_2^{(4)}E_2 + c_2^{(4)}H_2, \\ y_2^{(5)} &\sim a_2^{(5)} + b_2^{(5)}E_2 + c_2^{(5)}H_2, \\ y_2^{(6)} &\sim a_2^{(6)} + b_2^{(6)}E_2 + c_2^{(6)}H_2, \end{aligned}\right\} \tag{18}$$

where the a, b, and c's are constants which are non-zero. A solution for f_2 is then

$$f_2(\eta) = \frac{1}{(CM)^2}y_2^{(4)} + \frac{b_{11}}{\beta}y_2^{(5)} + y_2^{(6)} + Ay_2^{(2)} + By_2^{(3)},$$

where A and B are constants at our disposal to ensure $f_2 \sim b_{10}$ with exponentially small terms. The E_2 and η-terms must thus be annulled and so from (17) and (18)

$$B = -\frac{\beta^2}{\alpha(CM)^2}, \quad b_{11} = \frac{\beta}{b_2^{(5)}}\left[\frac{\beta^2 b_2^{(3)}}{\alpha(CM)^2} - \frac{b_2^{(4)}}{(CM)^2} - b_2^{(6)}\right], \tag{19}$$

which thus determines b_{11} (as a function of CM) and so

$$f_2(\eta) \sim b_{10} = A\beta + \left[\frac{a_2^{(4)}}{(CM)^2} + \frac{b_{11}}{\beta}a_2^{(5)} + a_2^{(6)} - \frac{\beta^2 a_2^{(3)}}{(CM)^2}\right], \tag{20}$$

where A is still an undetermined constant as in the zero-slip case: it is available to join this solution onto a leading edge one.

It was shown by Murray (1965) that $a_2^{(3)}, a_2^{(4)}, a_2^{(5)}, b_2^{(3)}, b_2^{(4)}, b_2^{(5)}$ are all non-zero, and so the construction of $f_2(\eta)$ is assured. Note that the $g_2(\eta)$ above is not that given by Murray (1965) since the b_{11} is different in the two cases.

Write
$$f_2(\eta) = {}_1f_2(\eta) + A\,{}_2f_2(\eta), \quad b_{10} = {}_1b_{10} + A\,{}_2b_{10}, \tag{21}$$
where
$$\left.\begin{aligned} {}_1f_2 &= \frac{1}{(CM)^2}y_2^{(4)} + \frac{b_{11}}{\beta}y_2^{(5)} + y_2^{(6)} - \frac{\beta^2}{\alpha(CM)^2}y_2^{(3)}, \\ {}_2f_2 &= \eta f_0' - f_0, \end{aligned}\right\} \tag{22}$$
and ${}_1b_{10}$ is the square bracket in (20) and ${}_2b_{10} = \beta$. The derivatives at $\eta = 0$ are
$$\left.\begin{aligned} &{}_1f_2''(0) = 0,\dagger \quad {}_2f_2''(0) = \alpha = 1{\cdot}32824 = f_0''(0), \\ &g_2''(0) = b_{11}\alpha/\beta, \quad f_0'''(0) = 0 = g_2'''(0). \end{aligned}\right\} \tag{23}$$

Thus, the construction of the asymptotic solution for large ξ as given by (8) is assured, at least up to the $O(\log \xi/\xi_3)$ term. In view of the verification of the zero-slip case by Murray (1965) it is likely that the solutions for the next terms in (8) are obtained in the same way from the higher-order terms in the zero-slip case with a similar assurance of analytical and numerical construction.

To illustrate the effect of a slip velocity at the plate it is helpful to separate out from the above solution the purely slip effects. We use the subscripts O and S to denote the no-slip case and the slip effects, respectively. Thus
$$\left.\begin{aligned} A &= (A)_O + (A)_S, \\ b_{11} &= (b_{11})_O + (b_{11})_S, \\ g_2(\eta) &= (g_2(\eta))_O + (g_2(\eta))_S, \\ f_2(\eta) &= (f_2(\eta))_O + (f_2(\eta))_S, \end{aligned}\right\} \tag{24}$$
where $(b_{11})_O$, $(g_2(\eta))_O$, $({}_1f_2(\eta))_O$, $({}_2f_2(\eta))_O$ are given by Murray (1965). From (15), (19), (20), (21), (22), and (24),
$$\left.\begin{aligned} (b_{11})_S &= [1/(CM)^2 - 1]\,(b_{11})_O - \beta b_2^{(6)}/b_2^{(5)} \\ (g_2(\eta))_S &= (b_{11})_S\,(g_2(\eta))_O, \\ (f_2(\eta))_S &= ({}_1f_2(\eta))_S + (A)_S({}_2f_2(\eta))_O, \\ ({}_1f_2(\eta))_S &= [1/(CM)^2 - 1]\,({}_1f_2(\eta))_O + [y_2^{(6)} - y_2^{(5)}b_2^{(6)}/b_2^{(5)}]. \end{aligned}\right\} \tag{25}$$

In the manner described by Murray (1965) it is a simple matter to compute $y_2^{(6)}$ and hence $b_2^{(6)}$.

3. Slip velocity and shear stress on the plate

The slip velocity given by (1) and the shear stress coefficient c_f require evaluation of
$$\frac{\partial u_1}{\partial y_1} = \frac{1}{4\xi_1^2}\frac{\partial^2\psi_1}{\partial\eta_1^2} = \left(\frac{U^3}{\nu}\right)^{\frac{1}{2}}\frac{1}{4\xi_1}\frac{\partial^2 f(\xi,\eta)}{\partial\eta^2}. \tag{26}$$

With c_f defined by
$$c_f = \nu\frac{\partial u_1}{\partial y_1}\Big/\tfrac{1}{2}U^2,$$

† Even without calculating $y_2^{(6)}$ its double derivative at $\eta = 0$ can be deduced by comparison with $y_2^{(5)}$ and $y_2^{(3)}$ which have been calculated.

the shear stress on the plate is given by the last equation evaluated at $\eta = 0$ which from (3), (8) and (26) is

$$c_f = \frac{1}{2}\left(\frac{\nu}{Ux_1}\right)^{\frac{1}{2}}\left[f_0''(0) + \frac{f_0'''(0)}{2X} + \frac{g_2''(0)\log X}{X^2} + \frac{f_2''(0)}{X^2} + \frac{g_2'''(0)\log X}{X^3} + \ldots\right],$$

where $$X = [\xi]_{\eta=0} = \frac{1}{CM}\left(\frac{U}{\nu}\right)^{\frac{1}{2}} x_1^{\frac{1}{2}} = \frac{1}{\lambda}\left(\frac{\nu}{U}\right)^{\frac{1}{2}} x_1^{\frac{1}{2}}.$$

From (23), (24) and (25) we can write c_f at $\eta = 0$ in the form

$$c_f = \tfrac{1}{2}\alpha\left(\frac{\nu}{Ux_1}\right)^{\frac{1}{2}}\left[1 + \frac{\lambda^2 U}{\beta\nu x_1}\{(b_{11})_O + (b_{11})_S\}\log\frac{1}{\lambda}\left(\frac{\nu x_1}{U}\right)^{\frac{1}{2}} + \{(A)_O + (A)_S\}\frac{\lambda^2 U}{\nu x_1}\right.$$
$$\left. + \text{terms of order smaller than } 1/x_1^{\frac{3}{2}}\right],$$

$$= \tfrac{1}{2}\alpha\left(\frac{\nu}{Ux_1}\right)^{\frac{1}{2}}\left[1 + \frac{\lambda^2 U}{\beta\nu x_1}\left\{\frac{b_2^{(6)}}{b_2^{(5)}}\beta - \frac{(b_{11})_O}{(CM)^2}\right\}\log\lambda\left(\frac{U}{\nu x_1}\right)^{\frac{1}{2}} + \{(A)_O + (A)_S\}\frac{\lambda^2 U}{\nu x_1} + \ldots\right], \tag{27}$$

with α from (14) and $(b_{11})_S$ from (25). The first term in (27) is the Blasius term. The slip velocity on the plate is given by

$$u_1|_{y_1=0} = \lambda\left.\frac{\partial u_1}{\partial y_1}\right|_{y_1=0} = \frac{\lambda U^2}{2\nu}c_f,$$

with c_f from (27).

In the notation of this paper Mirels (1952) and Bell (1955) using the boundary-layer form of the Oseen equations found

$$c_f = \frac{2}{\sqrt{\pi}}\left(\frac{\nu}{Ux_1}\right)^{\frac{1}{2}}\left[1 - \frac{\lambda^2}{2}\left(\frac{U}{\nu x_1}\right) + O\left(\left(\frac{U\lambda^2}{\nu x_1}\right)^2\right)\right],$$

and Laurmann (1961) using the full Oseen equations found

$$c_f = \frac{2}{\sqrt{\pi}}\left(\frac{\nu}{Ux_1}\right)^{\frac{1}{2}}\left[1 - \frac{\lambda^2}{2}\left(\frac{U}{\nu x_1}\right) + \frac{\lambda}{2\pi x_1}\left(\log\frac{Ux_1}{2\nu} - 0{\cdot}04\right) + \ldots\right],$$

For small λ, the correction to the Blasius term is $O(\lambda^2)$ according to Mirels (1952), and $O(\lambda)$ according to Laurmann (1961). However, using the full Navier–Stokes equations, (27) suggests that the true correction to the Blasius term may be of $O(\lambda^2\log\lambda)$, or, from (2), if small λ implies small M (the Mach number), the correction is more appropriately $O[(\nu/Ux_1)\log\lambda]$. Until A_S, which may be a function of CM, is determined, however, the $O[(\nu/Ux_1)\log\lambda]$ correction can only be a suggestion. Since the analysis is for incompressible viscous flow it is this result ($M \ll 1$) which may possibly be verified from experiment.

From (2), if the Mach number M is $O(1)$, $U\lambda/\nu$ is of $O(1)$ and in this case the correction from (27) is $O(\lambda\log\lambda)$ with the A_S term being of $O(\lambda)$. If M is large, $U\lambda/\nu$ is large and the correction thus becomes $O(\lambda^2\log\lambda)$ with the A_S term being of $O(\lambda^2)$. All of these differ from previous suggestions for the correction.

In conclusion, even though the value of A is unknown at this stage, the correction to the shear stress of the zero-slip case as given by (27), which is obtained from the full Navier–Stokes equations, suggest fundamentally different results to those obtained previously, and is perhaps an indication that the use of the Oseen equations, in boundary-layer form or otherwise, is *not* appropriate for a study of slip-phenomena of the type considered above. The now familiar fact that in flows past finite bodies the second (asymptotic) term from the solution of the Oseen equations is *not* in general the second term of the solution of the Navier–Stokes equation tends to substantiate this conclusion and the fact that the skin-friction correction involves a logarithm of the slip coefficient λ.

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